On the canonical bundle formula after Fujino–Mori

<u>Set up</u> $\backslash\mathbb{C}$

different notation than FM

(X,Δ) = log pair X normal var.
Δ $\mathbb{Q}$-divisor
(not nec effective)
$K_X+\Delta$ $\mathbb{Q}$-Cartier

$f:(X,\Delta)\longrightarrow Y_{\text{smooth}}$ klt fibration of Kod. dimension 0 if

$f_*\mathcal{O}_X=\mathcal{O}_Y$, (1) (X_η,Δ_η) is sub-klt
where $\eta\in Y$ generic pt

(2) $\kappa(F_\eta, K_{F_\eta}+\Delta_\eta)=0$

Given such fibration

$\hookrightarrow$ we can cook up a klt-trivial fibration as defined by Enrica

Let $b \in \mathbb{Z}_{>0}$ s.t. $f_* \mathcal{O}_X(b\underbrace{(K_X+\Delta)}_{\in \mathbb{Z}})^{**} \neq 0$

Prop (4.2)

$\exists!$ $\mathbb{Q}$-divisor $D \in \mathrm{Pic}(Y)_{\mathbb{Q}}$

$$\bigoplus_{i\geq 0} \mathcal{O}_Y(\lfloor ibD \rfloor) \simeq \bigoplus_{i\geq 0} f_* \mathcal{O}_X(\lfloor ib(K_X+\Delta) - f_i^* K_Y \rfloor)$$

The above isomorphism induces eq.

$$K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Y + D) + B^{\Delta}$$

where

rigidity of $B_{\geq 0}$ $\longrightarrow$ • $f_* \mathcal{O}_X(\lfloor i B^{\Delta}_{\geq 0} \rfloor) = \mathcal{O}_Y$

verticality of $B_{\leq 0}$ $\longrightarrow$ • $\mathrm{codim}_Y (f(B^{\Delta}_{\leq 0})) \geq 2$

Construction commutes with étale base changes

Rem $\Gamma := \Delta - B^{\Delta}$

$f : (X, \Gamma) \to Y$ klt-trivial fibrat.

Pf As Y smooth

$$\Rightarrow \quad f_* \mathcal{O}_X(cb(K_X+\Delta - f^*K_Y))^{**}$$

is Cartier for some $c>0$

$$f_* \mathcal{O}_X(cb(K_X+\Delta) - f^*K_Y)^{**} \simeq \mathcal{O}_Y(cbL) \quad \text{some } L \in \mathrm{Pic}(Y)_{\mathbb{Q}}$$

$$\Rightarrow \quad f^* \mathcal{O}_Y(cbL) \hookrightarrow \mathcal{O}_X(cb(K_X+\Delta - f^*K_Y))$$

on $X \setminus f^{-1}(\Sigma)$

where Σ has codim ≥ 2

$\Rightarrow \exists \ \Theta^\Delta$ s.t

$$bc(K_X+\Delta) = f^*(bc(L)) + \Theta^\Delta$$

You take $B^\Delta := \dfrac{\Theta^\Delta}{bc}$

$$K_X + \Delta = f^*L + B^\Delta$$

$$f : (X, \Gamma := \Delta - B^\Delta) \longrightarrow Y$$

<u>Recall</u> B_Y & M_Y

- For any prime P divisor

$$\gamma_P := \max\{t \in \mathbb{R} \mid (X, \Delta + t f^* P) \text{ is sub-lc over } \eta_P\}$$

Bertini theorem

↳ for all except finitely many P satisfy $\gamma_P = 1$

$$B_Y = \sum_{P \text{ prime} \subset Y} (1 - \gamma_P) P$$

(it depends on (X, Δ))

discriminant / boundary div.

$$M_Y = L. - K_Y - B_Y$$

moduli part

CBF $\quad K_X + \Gamma = f^*(K_Y + B_Y + M_Y)$

Aim
- study singularities of (Y, B_Y)
- positivity properties of M_Y

To do so, we pass to higher models of $f : X \to Y$ in a nice "snc format"

$$f : (X, \Delta) \longrightarrow Y$$

Hyp $\longrightarrow$ (X, Δ) is KLT

- $g : Z \longrightarrow X$ be a log resolution

  $K_Z + \Theta = g^*(K_X + \Delta)$ for which

  $\exists\ \Sigma \subset Y$ divisor s.t.

Θ^{hor} = horiz. part / Y (1) Θ^{hor} is snc over $Y \setminus \Sigma$

$h := f \circ g : Z \to Y$ (2) $h(\Theta^{v}) \subset \Sigma$

(3) f flat over $Y \setminus \Sigma$

- We resolve singularities of Σ :

  $\pi : Y' \to Y$ birational

  (1) $\pi^{-1}(\Sigma) = \Sigma'$ is snc

  (2) $Y' \setminus \Sigma' \simeq Y \setminus \Sigma$

  (3) $X_1 := (X \times_Y Y')^{\wedge}_{main}$

  flat over Y'

$$
\begin{array}{ccc}
Z & \longleftarrow & X' \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\
X & \xleftarrow{\ \tau\ } & (X_1)^{norm} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
Y & \xleftarrow[\ \pi\ ]{} & Y'
\end{array}
\qquad h = f\circ g : Z \to Y, \quad h' = f'\circ g' : X' \to Y'
$$

common resolution ⊛

s.t. ⊛ $K_{X'} + \Delta' = (g')^* \tau^* (K_X + \Delta)$,

- $(h')^{-1}(\Sigma') \cup \Delta'$ SNC
- $h'(\Delta'^{\,vert}) \subset \Sigma'$

Def We say that a birational model of f

$h' : (X', \Delta) \longrightarrow Y'$

satisfying the properties of the above diagram to be in FM-standard form.

Thm (CBF by Fujino-Mori)

Let $h': (X', \Delta') \to Y'$ a FM-model of (X, Δ)

$\exists$ $\mathbb{Q}$-divisor on X' s.t.

- $\Xi - \Delta'$ effective except $/X$
- (X', Ξ) is sub-klt

(existence guaranteed as (X, Δ) is klt)

Write as in 4.2: $K_{X'} + \Xi = (h')^*(K_{Y'} + B_{Y'} + M_{Y'}) + B^{\Xi}$

Then
1) $h'_* \mathcal{O}_{X'}(\lfloor i B^{\Xi}_{>0} \rfloor) = \mathcal{O}_{Y'}$
2) $B^{\Xi}_{\leq 0}$ g'-exceptional & $\operatorname{codim}(h'(B^{\Xi}_{\leq 0})) \geq 2$

$h': (Y', \Gamma' := \Xi - B^{\Xi}) \to Y$ is a klt trivial fibration

(3)

$$H^0(X, \lfloor ib(K_X+\Delta)\rfloor) = (Y', \lfloor ib(K_{Y'} + B_{Y'} + M_{Y'}))$$

(4) $M_{Y'}$ is nef

(5) Bound on the coefficients of B_Y

(~ adjunction)

Suppose (X', Ξ) is klt

Let $N>0$ s.t. $N(h')^* M_{Y'}$ & $N\,\Xi^\vee$ are $\mathbb{Z}$-div.

Then $\forall$ P prime divisor on Y'

$$\mathrm{coeff}_P\, B_{Y'} = 1 - \frac{n}{m}$$

& $0 < n \leq m$

Rem what is missing is a stabilisation / descent for the divisors

$K_Y + B_Y$ & M_Y.

(more next week)

Pf (1) & (2) is the properties of construction 4.2.

(3)

$$H^0(X, \lfloor ib(K_X+\Delta)\rfloor) \simeq H^0(X', \lfloor ib(K_{X'}+\Delta')\rfloor)$$

$$\underset{\substack{\Xi - \Delta \\ \text{eff \& exc}/X}}{\simeq} H^0(X', \lfloor ib(K_{X'}+\Xi)\rfloor)$$

$$\simeq H^0(X', \lfloor ib(K_{X'}+\Xi)\rfloor + \underbrace{ibB^{\Xi}_{\leq 0}}_{\text{contracted over } X_1^{\text{norm}}})$$

$$\simeq H^0(X', \lfloor (h')^* ib(K_{Y'}+B_{Y'}+M_{Y'}) + ibB^{\Xi}_{\leq 0}\rfloor)$$

$$\underset{\substack{\text{proj. form} \\ \& (1)}}{\simeq} H^0(Y', \lfloor ib(K_{Y'}+B_{Y'}+M_{Y'})\rfloor)$$

(4) we note that the moduli part does not depend on Ξ

We can assume $\Xi = \Delta^1_{\geq 0}$

Here we apply the following theorem because

(3) $h^1_* \mathcal{O}_{X^1}(\ulcorner B^{\Delta^1}_{\geq 0} \urcorner) = \mathcal{O}_{Y^1}$

(1) $h^1(\operatorname{Supp} \Delta^{1 \nu}) \subset \operatorname{Supp} \Sigma^1_1$

Thm (Kawamata semipositivity)

Let $f: X \to Y$ contraction of smooth projective varieties

Let $\Sigma \subset Y$ snc divisor s.t.

(nice snc form)

- $f^{-1}(\Sigma)_{red} \subset \Gamma = \sum D_j$ snc on X
- $f: X \setminus f^{-1}(\Sigma) \longrightarrow Y \setminus \Sigma$ smooth

Let $D = \sum d_j D_j$ (possibly negative) s.t.

(1) $f: \mathrm{Supp}(D^h) \longrightarrow Y$ is relatively snc over $Y \setminus \Sigma$

$f(\mathrm{Supp}\, D^v) \subset \Sigma$

klt condition → (2) $d_j < 1$ if D_j hori $\Leftrightarrow (X_\eta, D_\eta)$ sub klt

rigidity of negative horiz part of D → (3) $\dim_{k(\eta)} f_* \mathcal{O}_X(\lceil -D \rceil) \otimes k(\eta) = 1$

(4) $K_X + D \sim f^* L$

$\Rightarrow$ M_Y is nef

<u>Rem</u> This is the Hodge theoretic input for the CBF!

"Sketch of proof"

- one can reduce via semistable reduction & covering trick to recognize on $f'': X'' \to Y''$

$$M_{Y''} = f''_* \mathcal{O}_{X''}(\omega_{X''/Y''})$$

& f'' has also unipotent local monodromies

$\overset{\text{Hodge}}{\underset{\text{theory}}{\Longrightarrow}}$ $f''_* \mathcal{O}_X(\omega_{X''/Y''})$ is semipositive

(see it as a piece of filtration on a VHS)

(5) by cutting with hyperplanes, we can suppose $Y' =$ curve

<u>Note</u> (X', Ξ) klt $\Rightarrow B^{\Xi} \geq 0$
Y' curve

<u>Lemma</u> $K_{X'/Y'} + \Xi + (h^{-1}\Sigma')_{red}$
$$\geq h'^{*}(M_{Y'} + \Sigma')$$

Pf $B^{\Xi}_{<0} = 0$, then

$$K_{X'/Y'} + \Xi + (h^{-1}\Sigma)_{red}$$

def of γ $\to$ $\geq K_{X'/Y'} + \Xi - B^{\Xi} + (h')^{*}\sum_i \gamma_p P$

$$= h'^{*}(M_{Y'} + \Sigma')$$

we can prove bounds

Check

$$\mathcal{N}(K_{X',Y'} + \Xi) - h^* \mathcal{N} M_{Y'} + \mathcal{N}(h' \Sigma')_{red}$$
$$\geq \mathcal{N} h'^* M_{Y'}$$

$$\Rightarrow \mathcal{N}(h')^* B_{Y'} + \mathcal{N} B^{\Xi} + \mathcal{N}(h'^{-1} \Sigma')_{red}$$
$$\overset{(**)}{\geq} \mathcal{N} h^* \Sigma'$$

Now $h'^* P = \sum_k a_k F_k$

$P \in \operatorname{Supp} B_{Y'} \iff$

$$\operatorname{Supp}(\mathcal{N}(h')^* B_{Y'} - \mathcal{N}(1-\gamma_P) h'^* P)$$
$\not\supset F_c$ for some c

$$\overset{(**)}{\Longrightarrow} \mathcal{N}(1-\gamma_P) a_k \geq \mathcal{N} a_k$$

$\Rightarrow \quad (1-\gamma_p) \geq \frac{a_k - 1}{a_k} = 1 - \frac{1}{a_k}$

& $\quad (1-\gamma_p) \in \mathbb{Z}[\frac{1}{a_c}]$ for some c

$(\Rightarrow \quad (Y', B_{Y'})$ klt$)$ ■

MAIN APPLICATION

① FINITE GENERATION OF CANONICAL RING

Thm (X,Δ) klt proj.

$\kappa(K_X+\Delta) = \ell \geq 0$

$\Rightarrow \exists \ (Y, \Delta_Y)$ klt pair of $\dim = \ell$

$R(X, K_X+\Delta) \simeq R(Y, K_Y+\Delta_Y)$

up to truncation

Cor (after BCHM) canonical ring of klt pairs are f.g.

Pf $f : (X, \Delta) \dashrightarrow Z$ Iitaka fibration of $K_X + \Delta$

For canonical mg we can then suppose by Main Theorem that $f : (X, \Delta) \to Y$ is in FM-standard snc format.

By dimension reasons,

$K_Y + B_Y + M_Y$ big &

$R(X, \Delta) \underset{\text{trunc}}{\sim} R(Y, K_Y + B_Y + M_Y)$

$K_Y + B_Y + M_Y \sim_{\mathbb{Q}} \underset{\text{ample}}{A} + \underset{\text{effective}}{G}$ by Kodaira lemma

By (5), (Y, B_Y) klt

$\Rightarrow \exists\, \varepsilon > 0 \quad (Y, B_Y + \varepsilon G)$ klt

M_Y nef! $\underset{\text{crucial}}{\Rightarrow}$ $M + \varepsilon A$ ample $\sim_{\mathbb{Q}} \delta H^{\text{nice}}$, $\delta << 1$

$\Rightarrow (Y, B_Y + \varepsilon G + \delta H)$ is klt easy to check:

$$(n+m\varepsilon)(K_Y+B_Y+M_Y)$$
$$= n(K_Y+B_Y+M_Y) + m\varepsilon(A+G)$$
$$\sim n(K_Y+B_Y+\varepsilon G+\delta H)$$

so the canonical ring

$$R(Y, K_Y+B_Y+M_Y)$$

$$\underset{\text{up to truncation}}{\sim} R(Y, \underbrace{K_Y+B_Y+\varepsilon G+\delta H}_{\text{klt}})$$

■

<u>Cor</u> log canonical models of klt pairs have klt-type singularities.